

2. The sequence in the regeneration of a plate row is as follows: healing, stretching of the remaining part or parts of the canal and concentration of mesogleal cells at the wound area, fusion of parts of the canal, hollowing out at the region of fusion and formation of plates above the new canal.

3. Cells from the remaining part or parts of a row or from neighboring rows and also nonspecialized cells in the mesoglea take an active part in the regeneration of plate rows.

4. Contracting muscle strands which are anchored to the plate row and also in the mesoglea aid in the reformation of a row following its removal in part or as a whole.

\* Contribution No. 18 from the Department of Biology, Brooklyn College.

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Th. Mortensen, "On Regeneration in Ctenophores," *Vidensk. Meddel. fra Dansk naturh. Foren.*, **66**, 45 (1913).

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## IMMERSION OF THE FOURIER TRANSFORM IN A CONTINUOUS GROUP OF FUNCTIONAL TRANSFORMATIONS

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The Fourier transform  $g(u)$  of a function  $f(x)$  is defined by

$$g(u) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{iux} dx \quad (1)$$

where  $\int$  means integration from  $-\infty$  to  $+\infty$ . This possesses\* the inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int g(u) e^{-iux} dx. \quad (2)$$

Evidently if we write  $F$  for the operation performed on  $f$  in (1) then (1) and (2) can be written

$$g(u) = Ff(x), \quad f(-x) = Fg(u) = F^2f(x),$$

from which it follows that

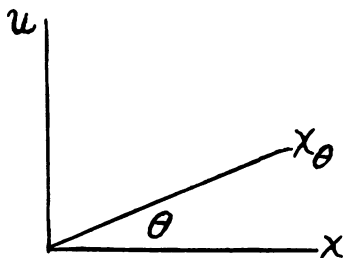
$$f(x) = F^4f(x) \quad (3)$$

so the transformation is of period 4. Another way of regarding (2) is that the operation there performed on  $g(u)$  is  $F^{-1}$  for this is consistent with

$$f(x) = F^{-1} g(u) = F^{-1} F f(x). \quad (4)$$

Hence the operation  $F$  generates a cyclic group of order 4 which is isomorphic with the group of rotations of a plane about a fixed point through multiples of a right angle. Now every continuous group of transformations is generated by an Hermitian operator, and conversely every Hermitian operator generates a group of unitary transformations.<sup>†</sup> Hence there exists a continuous group of functional transformations containing the ordinary Fourier transforms as a subgroup. In this paper the continuous group is explicitly found. It will not, however, be necessary to make further reference in the work that follows to the general immersion theory.

It is convenient to introduce a group space as shown in the figure in which the notation  $x_\theta$  is assigned to the argument of a function which is generated out of  $f(x)$  by application to it of the functional transformation  $F_\theta$ . In this notation  $F_\theta f(x)$  will be a transformed function of the argument  $x_\theta$  which we may write as  $f_\theta(x_\theta)$ . Evidently  $x$  is  $x_0$  and  $f(x)$  is more explicitly  $f_0(x_0)$  in this notation.



Likewise the ordinary Fourier transform becomes  $F_{\pi/2}$  and we have  $u = x_{\pi/2}$  and

$$g(u) = f_{\pi/2}(x_{\pi/2}).$$

The kernel of the integral which represents the operation  $F$  will depend on two variables: one is  $x_\alpha$  where the operation is applied to a function  $f_\alpha(x_\alpha)$ ; the other is  $x_{\alpha+\theta}$  corresponding to the fact that the operation "rotates" the function through an angle  $\theta$  in the group space. The general integral representation of (1) becomes in this notation

$$f_{\alpha+\theta}(x_{\alpha+\theta}) = \int K_\theta(x_\alpha, x_{\alpha+\theta}) f_\alpha(x_\alpha) dx_\alpha. \quad (5)$$

The problem is that of finding the kernel which represents the operation  $F_\theta$ . Before proceeding to this question we may observe that the kernel will be necessarily singular for  $\theta = 0$  and  $\theta = \pi$ . For  $\theta = 0$  the operation should reduce to the identical transformation

$$f_\alpha(x') = \int K_0(x_\alpha, x'_\alpha) f_\alpha(x_\alpha) dx_\alpha \quad (6)$$

which can only be forced into this form by the introduction of an improper function  $\delta(x)$  defined by the properties

$$\int \delta(x) dx = 1 \text{ and } \delta(x) = 0 \text{ for } x \neq 0. \quad (7)$$

In terms of  $\delta(x)$  we have

$$K_0(x_\alpha, x'_\alpha) = \delta(x_\alpha - x'_\alpha). \quad (8)$$

Similarly for  $\theta = \pi$  we must have, since  $x_{\alpha+\pi} = -x_\alpha$

$$f_{\alpha+\pi}(x_{\alpha+\pi}) = f_\alpha(-x_\alpha)$$

so the kernel in this case also becomes the improper  $\delta(x)$

$$K_\pi(x_\alpha, x_{\alpha+\pi}) = \delta(x_\alpha + x_{\alpha+\pi}). \quad (9)$$

Otherwise one has no reason to expect singular behavior of the kernel. In particular from the ordinary theory of Fourier transforms,

$$\begin{aligned} K_{\pi/2}(x_\alpha, x_{\alpha+\pi/2}) &= \frac{1}{\sqrt{2\pi}} \exp \{i x_\alpha x_{\alpha+\pi/2}\} \\ K_{3\pi/2}(x_\alpha, x_{\alpha+3\pi/2}) &= \frac{1}{\sqrt{2\pi}} \exp \{-i x_\alpha x_{\alpha+3\pi/2}\}. \end{aligned} \quad (10)$$

An important property of the Fourier transform which it is desirable to preserve is

$$\int |g(u)|^2 du = \int |f(x)|^2 dx$$

which in our notation becomes

$$\int |f_\theta(x_\theta)|^2 dx_\theta = \int |f(x)|^2 dx \quad (11)$$

for all  $\theta$ . Proceeding formally we have

$$\int |f_\theta(x_\theta)|^2 dx_\theta = \int \int \int f(x) K_\theta(x, x_\theta) \overline{f(x')} \overline{K_\theta(x', x_\theta)} dx dx' dx_\theta$$

which will be true if

$$\int K_\theta(x, x_\theta) \overline{K_\theta(x', x_\theta)} dx_\theta = \delta(x - x'). \quad (12)$$

The law of combination of transformations is given by:

$$\begin{aligned} f_{\theta+\varphi}(x_{\theta+\varphi}) &= \int f_\theta(x_\theta) K_\varphi(x_\theta, x_{\theta+\varphi}) dx_\theta \\ &= \int \int f(x) K_\theta(x, x_\theta) K_\varphi(x_\theta, x_{\theta+\varphi}) dx dx_\theta \\ &= \int f(x) K_{\theta+\varphi}(x, x_{\theta+\varphi}) dx. \end{aligned}$$

The last two expressions are equivalent if

$$K_{\theta+\varphi}(x, x_{\theta+\varphi}) = \int K_\theta(x, x_\theta) K_\varphi(x_\theta, x_{\theta+\varphi}) dx_\theta. \quad (13)$$

The transformation with parameter  $-\theta$  must be reciprocal to that with parameter  $\theta$  so

$$K_{\theta-\theta}(x, x') = \delta(x - x') = \int K_\theta(x, x_\theta) K_{-\theta}(x_\theta, x') dx_\theta.$$

This will be consistent with (12) if we require of the kernels that

$$K_{-\theta}(x_\theta, x') = \overline{K_\theta(x', x_\theta)}. \quad (14)$$

We may find the desired kernel as follows. With each variable  $x_\theta$  we associate a Hermitian operator  $X_\theta$ . In particular write  $X_0 = X$  and  $X_{\pi/2} = U$  and demand that the operators  $X$  and  $U$  obey the commutation law

$$UX - XU = -i. \quad (15)$$

For application to functions of  $x$  the operators  $X$  and  $U$  can be explicitly represented by

$$Xf(x) = xf(x) \quad Uf(x) = -if'(x). \quad (16)$$

We then define the operator for  $X_\theta$  in this same representation as being the same combination of  $X$  and  $U$  as holds for  $x_\theta$  in terms of  $x$  and  $u$  regarded as ordinary coördinates in the group space,

$$X_\theta = sU + cX \quad (17)$$

where  $s = \sin \theta$  and  $c = \cos \theta$ .

The proper functions  $P(x, x')$  of the operator  $X$  satisfy

$$XP(x, x') = x'P(x, x') \text{ or } (x-x')P(x, x') = 0.$$

Hence  $P(x, x') = 0$  for  $x \neq x'$  and so with a suitable normalization  $P(x, x')$  corresponds to the kernel for the identical transformation. Similarly the proper function for  $U$ , say  $Q(x, u)$  satisfies

$$-i \frac{\partial}{\partial x} Q(x, u) = uQ(x, u)$$

which gives  $Q(x, u) = e^{iux}$  which if properly normalized gives the kernel of the ordinary Fourier transform. These facts suggest that the proper functions of  $X_\theta$  in this representation will give the general kernel  $K_\theta(x, x_\theta)$  after a suitable normalization. These proper functions satisfy

$$\left(-is \frac{\partial}{\partial x} + cx\right) K_\theta(x, x_\theta) = x_\theta K_\theta(x, x_\theta)$$

which gives

$$K_\theta(x, x_\theta) = \exp \left\{ -\frac{ic}{2s} x^2 + \frac{ixx_\theta}{s} \right\}$$

to within a factor which is an arbitrary function of  $x_\theta$ . Since  $c(\theta) = c(-\theta)$  and  $s(\theta) = -s(-\theta)$ , equation (14) determines the dependence on  $x_\theta$  so we have except for a normalizing constant  $C$ ,

$$K_{\theta}(x, x_{\theta}) = C \exp \left\{ -\frac{ic}{2s} x^2 + \frac{ixx_{\theta}}{s} - \frac{ic}{2s} x_{\theta}^2 \right\}.$$

The magnitude of  $C$  has to be determined by (12) which yields the equation

$$|C|^2 \exp \left\{ -\frac{ic}{2s} (x^2 - x'^2) \right\} \int \exp \left\{ \frac{i(x-x')x_{\theta}}{s} \right\} dx_{\theta} = \delta(x-x').$$

The integral here is divergent but is the same one which is interpreted as having the value  $2\pi s \delta(x-x')$  when this formalism is applied to the ordinary Fourier transform. Hence in a formal sense the kernel has the property (12) provided that

$$|C| = (2\pi s)^{-1/2}.$$

Finally we have to verify that the kernel has the group property expressed in (13). In doing this we make the restriction that neither  $\theta$  nor  $\varphi$  nor  $\varphi + \theta$  is equal to 0,  $\pi/2$ ,  $\pi$  or  $3\pi/2$ . Writing  $C = (2\pi s)^{-1/2} e^{i\delta}$  and  $c'$  and  $s'$  for  $\cos \varphi$  and  $\sin \varphi$ , we have to compute

$$\frac{e^{i2\delta}}{2\pi\sqrt{ss'}} \int \exp \left\{ -\frac{ic}{2s} x^2 + \frac{ixx_{\theta}}{s} - \frac{ic}{2s} x_{\theta}^2 - \frac{ic'}{2s'} x_{\theta}^2 + \frac{ix_{\theta}x_{\theta+\varphi}}{s'} - \frac{ic'}{2s'} x_{\theta+\varphi}^2 \right\} dx_{\theta}.$$

The integral may be computed by elementary methods giving the result

$$\frac{e^{i(2\delta \mp \pi/4)}}{\sqrt{2\pi s''}} \exp \left\{ -\frac{ic''}{2s''} x^2 + \frac{ixx_{\theta+\varphi}}{s''} - \frac{ic''}{2s''} x_{\theta+\varphi}^2 \right\}$$

where the ambiguous sign is  $-$  if  $s''/ss' > 0$  and  $+$  if  $s''/ss' < 0$  and  $c''$  and  $s''$  are written for  $\cos(\theta + \varphi)$  and  $\sin(\theta + \varphi)$ . This will be exactly equal to  $K_{\theta+\varphi}(x, x_{\theta+\varphi})$  if we choose  $\delta = \pi/4$  for angles in the first two quadrants and  $\delta = 3\pi/4$  for angles in the third and fourth quadrants. Hence the kernel is completely determined for angles other than 0 or  $\pi$  to be

$$K_{\theta}(x, x_{\theta}) = \frac{e^{i\delta}}{\sqrt{2\pi s}} \exp \left\{ -\frac{ic}{2s} x^2 + \frac{ixx_{\theta}}{s} - \frac{ic}{2s} x_{\theta}^2 \right\}. \quad (18)$$

For  $\theta = 0$  we define the functional transformation to be the identity. For  $\theta = \pi$  we define it as

$$f_{\alpha+\pi}(x_{\alpha+\pi}) = e^{i\pi/2} f_{\alpha}(-x_{\alpha+\pi}),$$

the factor  $e^{i\pi/2}$  being introduced for a reason which appears in the next paragraph.

We observe that with the values of  $\delta$  as determined in the last paragraph the transformation for  $\theta = \pi/2$  differs from the ordinary Fourier transform only because of the inclusion of the factor  $e^{i\pi/4}$ . This determination of the phase has the property both for the ordinary Fourier transform and for our generalization that if a succession of transforms be made such that their parameter values,  $\theta_1, \theta_2, \theta_3, \dots$  add up to  $2\pi$ , we do not get the identity but get instead the original function multiplied by  $-1$ . The representation of the group that has been found is thus a double-valued one and does not reproduce the identity until the sum of the parameter values of the successive transforms is equal to  $4\pi$ . It is evident that the factor introduced in the special definition of the transform for  $\theta = \pi$  is just what is needed to be in accord with these properties.

It will be of interest also to examine the behavior of  $f_\theta(x_\theta)$  as  $\theta \rightarrow 0$ . For  $\theta = 0$  the transformation gives the identity, but for  $\theta$  small  $f_\theta(x_\theta)$  is rather different from  $f(x)$ , as may be seen from a qualitative inspection of the properties of the kernel (18). Writing  $i\Phi$  for the argument of the exponential function in (18) we see that for  $\theta$  small the coefficients of  $x^2$ ,  $xx_\theta$  and  $x_\theta^2$ , will be large. Hence the kernel will be a rapidly oscillating function of  $x$  except in the neighborhood of  $x = x_\theta/c$  where  $\frac{\partial \Phi}{\partial x}$  vanishes.

Hence if  $f(x)$  is a smooth function the chief contributions to  $f_\theta(x_\theta)$  will come from the values of  $f(x)$  for  $x$  near to  $x_\theta/c$ . To bring out this fact we may write  $\Phi$  in the form

$$\Phi = -\frac{c}{2s}(x - x_\theta/c)^2 + \left(\frac{1}{sc} - \frac{c}{2s}\right)x_\theta^2.$$

The second term is independent of  $x$  and therefore  $f_\theta(x_\theta)$  will contain a factor  $\exp \left\{ i \left( \frac{1}{sc} - \frac{c}{2s} \right) x_\theta^2 \right\}$ . The first term has rapid oscillations except

near  $x = x_\theta/c$ . The zeros of  $\cos \frac{c}{2s}(x - x_\theta/c)^2$  that are nearest to  $x_\theta/c$  occur

for  $x = x_\theta/c \pm \sqrt{\pi s/c}$  and beyond these the kernel oscillates rapidly. Hence for  $\theta$  small and  $f(x)$  smooth the integral of  $f(x)$  occurring in the transform will be very roughly of the order of

$$Av[f(x)] \cdot \exp \left\{ i \left( \frac{1}{sc} - \frac{c}{2s} \right) x_\theta^2 \right\}$$

where  $Av[f(x)]$  is an average of  $f(x)$  over a range of the order  $x_\theta/c - \sqrt{\pi s/c}$  to  $x_\theta/c + \sqrt{\pi s/c}$ . As  $\theta \rightarrow 0$  this approaches the value  $f(x_\theta)$  so  $f_\theta(x_\theta)$  becomes qualitatively equal to

$$e^{i(1/sc - c/2s)x_\theta^2} f(x_\theta).$$

The exponential factor, however, becomes a highly singular function of  $x_\theta$  as  $\theta \rightarrow 0$  so the functional transform does not continuously approach the identity. However, it does have the property that

$$|f_\theta(x_\theta)| \rightarrow |f(x)|$$

as  $\theta \rightarrow 0$  for smooth functions.

It is a pleasure to acknowledge my indebtedness to helpful conversations with several of my colleagues, especially Professors J. von Neumann, H. Bohnenblust and S. Bochner.

\* See, for example, N. Wiener, *The Fourier Integral*, Cambridge University Press, p. 69 (1933).

† Stone, M. H., *Proc. Nat. Acad. Sci.*, **16**, 173 (1930); *Annals of Math.*, **33**, 643 (1932).

## LATTICES AND BICOMPACT SPACES

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In this note there are announced certain new results concerning the interrelations between distributive lattices and topological spaces.

**THEOREM 1.** *Given a distributive lattice  $L$  with zero and unit, there is a bicomact  $T_1$ -space  $S$ , a basis for whose closed sets is a lattice-homomorphic image of  $L$ .*

**COROLLARY.** *This basis for the closed sets of  $S$  is isomorphic to  $L$  if and only if  $L$  has the property that if  $a$  and  $b$  are different elements of  $L$  there exists an element  $c$  of  $L$  such that one of  $ac$  and  $bc$  is zero and the other is not zero.*

**THEOREM 2.** *If  $R$  is a  $T_1$ -space then the bicomact  $T_1$ -space  $S$  obtained by applying the process of Theorem 1 to the lattice of the closed sets of  $R$  is such that  $S$  contains a dense subset  $R'$  homeomorphic to  $R$ ; if  $F$  and  $G$  are closed disjoint subsets of  $R$ ,  $\bar{F}$  and  $\bar{G}$  their correspondents in  $S$  under the homeomorphism  $R \rightleftharpoons R'$ ,  $\bar{F}$  and  $\bar{G}$  the closures in  $S$  of  $F$  and  $G$ , then  $\bar{F}$  and  $\bar{G}$  are also disjoint; in the sense of Čech, the homology theory of  $R$  is identical with that of  $S$  and dimension  $R = \text{dimension } S$ .*

**SPECIAL CASE.**<sup>1</sup>  *$S$  is a Hausdorff space (and hence normal) if and only if  $R$  is a normal space.*<sup>2</sup>

The method is as follows. Let  $L$  be a distributive lattice with zero and unit. A collection  $C$  of elements of  $L$  with the property that the intersection of any finite number of elements of the collection is not zero, while  $C$  is a proper sub-collection of no collection having the same property, will